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Abstract

We study the stochastic parallel dynamics of Ising spin systems defined on finitely connected directed random graphs with arbitrary degree distributions, using generating functional analysis. For fully asymmetric graphs the dynamics of the system can be completely solved, due to the asymptotic absence of loops. For arbitrary graph symmetry, we solve the dynamics exactly for the first few time steps, and we construct approximate stationary solutions.

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1. Introduction

Problems defined on finitely connected (partially) random graphs have been studied intensively in various fields of science. Examples from physics are models of spin glasses and related magnetic systems [1–10]. A second field is information theory and computer science, where calculations involving finitely connected random graphs emerge in the context of error correcting codes [11–18], lossy data compression [19–26], CDMA multi-user detection [27–29] and combinatorial optimization [30–34]. In biology and the social and economical sciences one finds processes on finitely connected random graphs in the context of neural [35, 36] as well as proteomic and gene regulation networks, and small world models [37–40]. In all these fields, techniques of statistical mechanics have been decisive in making progress. The initial focus of research has been on establishing the equilibrium properties of processes on finitely connected random graphs, mainly by applying replica theory and the cavity method. In a second wave also the dynamical properties of processes on finitely connected random graphs have begun to be investigated in detail, see e.g. [41–50]. In real-world applications involving finitely connected random graphs, there are many situations in which it is essential that

dynamical properties be understood quantitatively, and this is especially true for technological applications, such as the decoding of error correcting codes and the detection dynamics of CDMA multi-user detectors, and for biological information processing systems where the absence of detailed balance rules out equilibrium methods.

In most of the problems described above, the dynamical variables and their interactions are represented as nodes and edges of finitely connected graphs, respectively. The simplest such graphs are Poissonian ones, i.e. sparse Erdös–Rényi graphs with a Poissonnian degree distribution in the thermodynamic limit. However, the broad spectrum of problems to be understood demands dynamical techniques that are able to treat more general random graph ensembles. The dynamics of processes on graphs drawn from ensembles with arbitrary degree distributions have so far been studied using dynamical replica theory [45, 48] and the cavity method [50]. One particular dynamical formalism, which enjoys the appeal of full exactness (if it applies), is the generating functional method of De Dominicis [51]. This method was first used by Hatchett *et al* to investigate the parallel dynamics of bond-disordered Ising spin systems on finitely connected random graphs [44], but their study was limited to finitely connected Poissonian random graphs. In information theoretic problems and modeling real-world systems, there are many examples which are characterized by finitely connected random graphs with specific degree distributions. For example, a power-law distribution is often utilized to discuss 'small-world' networks. In the framework of error correcting codes, a one-point distribution and specific (complex) degree distributions which give higher performance are applied.

In this paper we generalize the analysis of [44]: we use generating functional techniques to analyze the parallel stochastic dynamics of Ising spin models defined on finitely connected random graphs, which are drawn from ensembles in which the degree distributions can be chosen arbitrarily (including a power-low distribution, a one-point distribution and so on). We first derive general dynamical order parameter equations, which give a transparent exact description of collective processes in finitely connected Ising systems. We then consider the simplest case of fully asymmetric finitely connected random graphs. Here the theory acquires a simple form due to the absence of the loops, and the dynamics becomes very simple. Away from fully asymmetric graphs, we calculate the first few time steps exactly, and construct (for fully symmetric graphs) approximate equilibrium solutions. To confirm the validity of our theory, we present numerical results for some typical conditions and typical graph ensembles; although our theory can handle arbitrary degree distributions, our examples are mostly regular random graphs (for simplicity).

2. Model definitions

Let us consider a system of N Ising-type spins placed on the edges of a finitely connected directed random graph with an as yet arbitrary degree distribution. The dynamics are given by a Markov process which represents synchronous stochastic spin alignment to local fields. The probability $P_t(\boldsymbol{\sigma})$ of finding the microscopic state $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$ at time t can be written as

$$P_t(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}' \in \{-1,1\}^N} W_{t-1}[\boldsymbol{\sigma}|\boldsymbol{\sigma}'] P_{t-1}(\boldsymbol{\sigma}'), \tag{1}$$

$$W_{t-1}[\boldsymbol{\sigma}|\boldsymbol{\sigma}'] = \prod_{i=1}^{N} \frac{\mathrm{e}^{\beta \sigma_i h_i(\boldsymbol{\sigma}', t-1)}}{2 \cosh[\beta h_i(\boldsymbol{\sigma}', t-1)]} \tag{2}$$

where $h_i(\boldsymbol{\sigma}, t)$ are local fields, defined as

$$h_i(\boldsymbol{\sigma}, t) = \frac{1}{c} \sum_{\substack{j \neq i}}^{N} c_{ij} J_{ij} \sigma_j + \theta_i(t). \tag{3}$$

They involve a connectivity matrix $\boldsymbol{c} = \{c_{ij}\} \in \{0, 1\}^{N \times N}$, whose entries specify which spins interact, and define a (generally directed) random graph. The parameters $\theta_i(t) \in \mathbb{R}$ are time-dependent external fields, and $c > 0$. The bond strengths J_{ij} are symmetric, i.e. $J_{ij} = J_{ji}$, and are drawn independently from a bond distribution $\tilde{p}(J)$. We define the degree k_i of node i (i.e. the number of links to the node i) as

$$k_i = \sum_{j=1}^{N} c_{ij} \tag{4}$$

(this would often be called the 'in-degree'). The entries of the matrix $\boldsymbol{c}$ are chosen randomly according to the following connectivity distribution, in which all N degrees k_i are constrained:

$$\hat{p}(\boldsymbol{c}) = \frac{\left(\prod_{i=1}^{N} \prod_{j>i}^{N} \hat{p}(c_{ij}) \hat{p}(c_{ji}|c_{ij}) \right) \left(\prod_{i=1}^{N} \delta_{k_i, \sum_{j=1}^{N} c_{ij}} \right)}{\sum_{\boldsymbol{c}'} \left(\prod_{i=1}^{N} \prod_{j>i}^{N} \hat{p}(c'_{ij}) \hat{p}(c'_{ji}|c'_{ij}) \right) \left(\prod_{i=1}^{N} \delta_{k_i, \sum_{j=1}^{N} c'_{ij}} \right)}, \tag{5}$$

where

$$\hat{p}(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N} \right) \delta_{c_{ij},0}, \tag{6}$$

$$\hat{p}(c_{ji}|c_{ij}) = \varepsilon \delta_{c_{ji},c_{ij}} + (1 - \varepsilon) \left[\frac{c}{N} \delta_{c_{ji},1} + \left(1 - \frac{c}{N} \right) \delta_{c_{ji},0} \right], \tag{7}$$

(with the Kronecker symbol $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = 0$ otherwise). All k_i are randomly and independently drawn from a given degree distribution $p(k)$, with $\sum_k k p(k) = c$, so $c = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij}$. This procedure generates an ensemble of random graphs, with a prescribed degree distribution $p(k)$ and a parameter ε that controls the symmetry of the graph. For $\varepsilon = 1$, the connectivity becomes symmetric. In the remainder of this paper we will write averages over the bond variables $\{J_{ij}\}$ as $\langle \cdots \rangle_J = \int dJ \, \tilde{p}(J)(\cdots)$ and averages over both the microscopic graph and bond variables, i.e. over $\{c_{ij}, J_{ij}\}$ as $\overline{[\cdots]}$.

3. Generating functional analysis

3.1. The disorder-averaged generating functional

We follow the approach of [44], but apply it to the present generalized spin model. We assume that the macroscopic behavior of our system depends only on the statistical properties of the disorder. The joint distribution of any trajectory $\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(t_m)$ is given by products of the individual transition probabilities of the Markov chain:

$$P[\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(t_m)] = p_0[\boldsymbol{\sigma}(0)] \prod_{t=0}^{t_m-1} W[\boldsymbol{\sigma}(t+1)|\boldsymbol{\sigma}(t)], \tag{8}$$

where $p_0[\boldsymbol{\sigma}(0)]$ represents the initial conditions. The generating functional $Z[\boldsymbol{\psi}]$ for the process is now defined as [51]

$$\overline{Z[\boldsymbol{\psi}]} = \overline{\left\langle \exp\left[-i \sum_{i=1}^{N} \sum_{t=0}^{t_m} \psi_i(t) \sigma_i(t) \right] \right\rangle} \tag{9}$$

where $\langle\cdots\rangle$ denotes averaging over the microscopic process, namely

$$\langle\cdots\rangle = \sum_{\boldsymbol{\sigma}(0)\in\{-1,1\}^N}\cdots\sum_{\boldsymbol{\sigma}(t_m)\in\{-1,1\}^N} p[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_m)](\cdots), \tag{10}$$

and $\boldsymbol{\psi} = (\psi_i(t))$ denote the generating fields. We isolate the local fields at all stages by inserting appropriate integrals over integral representations of the Dirac delta function:

$$1 = \int\{\mathrm{d}\boldsymbol{h}\,\mathrm{d}\hat{\boldsymbol{h}}\}\prod_{i=1}^{N}\prod_{t=0}^{t_m-1}\exp\left[\mathrm{i}\hat{h}_i(t)\left(h_i(t)-\frac{1}{c}\sum_{j\neq i}^{N}c_{ij}J_{ij}\sigma_j(t)-\theta_i(t)\right)\right], \tag{11}$$

where $\{\mathrm{d}\boldsymbol{h}\,\mathrm{d}\hat{\boldsymbol{h}}\} = \prod_{i=1}^{N}\prod_{t=0}^{t_m-1}[\mathrm{d}h_i(t)\,\mathrm{d}\hat{h}_i(t)/2\pi]$. The generating functional then takes the form

$$\overline{Z[\boldsymbol{\psi}]} = \int\{\mathrm{d}\boldsymbol{h}\,\mathrm{d}\hat{\boldsymbol{h}}\}\sum_{\boldsymbol{\sigma}(0)}\cdots\sum_{\boldsymbol{\sigma}(t_m)}p_0[\boldsymbol{\sigma}(0)]\overline{\exp\left[-\frac{\mathrm{i}}{c}\sum_{i=1}^{N}\sum_{t=0}^{t_m-1}\hat{h}_i(t)\sum_{j\neq i}^{N}c_{ij}J_{ij}\sigma_j(t)\right]}$$

$$\times\prod_{i=1}^{N}\prod_{t=0}^{t_m-1}\mathrm{e}^{\mathrm{i}\hat{h}_i(t)[h_i(t)-\theta_i(t)]-\mathrm{i}\psi_i(t)\sigma_i(t)+\beta\sigma_i(t+1)h_i(t)-\ln 2\cosh[\beta h_i(t)]}. \tag{12}$$

Hereafter, we will change our notation from $\boldsymbol{\sigma}(t) = (\sigma_1(t),\ldots,\sigma_N(t))$ (the N-spin configuration at time t) to $\boldsymbol{\sigma}_i = (\sigma_i(0),\ldots,\sigma_i(t_m))$ (the path taken from $t = 0$ to $t = t_m$ by spin i). Similarly, we define single-site paths for external, local and conjugate fields, namely $\boldsymbol{\theta}_i = (\theta_i(0),\ldots,\theta_i(t_m))$, $\hat{\boldsymbol{h}}_i = (\hat{h}_i(0),\ldots,\hat{h}_i(t_m))$, etc.

Using the integral form of the Kronecker delta to represent the degree constraints

$$\delta_{k_i,\sum_{j=1}^{N}c_{ij}} = \int_0^{2\pi}\frac{\mathrm{d}\omega_i}{2\pi}\exp\left[\mathrm{i}\omega_i\left(k_i-\sum_{j=1}^{N}c_{ij}\right)\right] \tag{13}$$

the term in (12) containing the disorder $\{c_{ij}, J_{ij}\}$ becomes

$$\overline{\exp\left[-\frac{\mathrm{i}}{c}\sum_{i=1}^{N}\sum_{t=0}^{t_m-1}\hat{h}_i(t)\sum_{j\neq i}^{N}c_{ij}J_{ij}\sigma_j(t)\right]}$$

$$= \frac{1}{Z_c}\left(\prod_{i=1}^{N}\int\frac{\mathrm{d}\omega_i}{2\pi}\,\mathrm{e}^{\mathrm{i}\omega_i k_i}\right)\exp\left[\frac{c}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\varepsilon\,\mathrm{e}^{-\frac{\mathrm{i}J}{c}(\boldsymbol{\sigma}_i\cdot\hat{\boldsymbol{h}}_j+\boldsymbol{\sigma}_j\cdot\hat{\boldsymbol{h}}_i)-\mathrm{i}(\omega_i+\omega_j)}\right.$$

$$\left.+ (1-\varepsilon)\,\mathrm{e}^{-\frac{\mathrm{i}J}{c}\boldsymbol{\sigma}_i\cdot\hat{\boldsymbol{h}}_j-\mathrm{i}\omega_j} + (1-\varepsilon)\,\mathrm{e}^{-\frac{\mathrm{i}J}{c}\boldsymbol{\sigma}_j\cdot\hat{\boldsymbol{h}}_i-\mathrm{i}\omega_i} + \varepsilon - 2\rangle_J\right], \tag{14}$$

where Z_c denotes the normalization constant of the connectivity distribution $\hat{p}(\boldsymbol{c})$ in (5), i.e. $Z_c = \sum_{\boldsymbol{c}}\left(\prod_{i=1}^{N}\prod_{j>i}^{N}\hat{p}(c_{ij})\hat{p}(c_{ji}|c_{ij})\right)\left(\prod_{i=1}^{N}\delta_{k_i,\sum_{j=1}^{N}c_{ij}}\right)$. Details on the calculation of (14) can be found in appendix A. To achieve site factorization we choose factorized homogeneous initial conditions, i.e. $p_0[\boldsymbol{\sigma}(0)] = \prod_i p_0[\sigma_i(0)]$, and we introduce the following order parameter functions:

$$P(\boldsymbol{\sigma},\hat{\boldsymbol{h}}) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\hat{\boldsymbol{h}}-\hat{\boldsymbol{h}}_i), \tag{15}$$

$$Q(\boldsymbol{\sigma},\hat{\boldsymbol{h}}) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}\delta(\hat{\boldsymbol{h}}-\hat{\boldsymbol{h}}_i)\,\mathrm{e}^{-\mathrm{i}\omega_i}. \tag{16}$$

It should be noted that one could also have used the order parameter function $R(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}, \omega) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\hat{\boldsymbol{h}} - \hat{\boldsymbol{h}}_i) \delta(\omega - \omega_i)$ to achieve site factorization in the generating functional. The reason for us to work with (15) and (16) is that the latter will have transparent physical interpretations. We determine the initial state $\sigma_i(0)$ according to the distribution $p_0[\sigma_i(0)] = \frac{1}{2}[1 + \sigma_i(0)m(0)]$, where $m(0)$ denotes the initial magnetization. To introduce the two-order parameter functions in the generating functional, we use the integral form of Dirac's delta function and insert

$$1 = \int \{dP \, d\hat{P}\} \exp\left[iN \sum_{\boldsymbol{\sigma}} \int d\hat{\boldsymbol{h}} \, \hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) \left(P(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) - \frac{1}{N} \sum_{i=1}^{N} \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\hat{\boldsymbol{h}} - \hat{\boldsymbol{h}}_i)\right)\right], \quad (17)$$

$$1 = \int \{dQ \, d\hat{Q}\} \exp\left[iN \sum_{\boldsymbol{\sigma}} \int d\hat{\boldsymbol{h}} \, \hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) \left(Q(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) - \frac{1}{N} \sum_{i=1}^{N} \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\hat{\boldsymbol{h}} - \hat{\boldsymbol{h}}_i) \, e^{-i\omega_i}\right)\right], \quad (18)$$

where $\{dP \, d\hat{P}\} = \prod_{\boldsymbol{\sigma}, \hat{h}}[dP(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})d\hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})N/2\pi]$, and we choose the equivalent definition for the short-hand $\{dQ \, d\hat{Q}\}$. We then have

$$\overline{Z[\boldsymbol{\psi}]} = \frac{1}{Z_c} \int \{dP \, d\hat{P} \, dQ \, d\hat{Q}\} \, e^{N\Psi[\{P, \hat{P}, Q, \hat{Q}\}]}, \quad (19)$$

where

$$\Psi[\{P, \hat{P}, Q, \hat{Q}\}] = \frac{c(\varepsilon - 2)}{2} + \frac{c}{2} \sum_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} \int d\hat{\boldsymbol{h}} \, d\hat{\boldsymbol{h}}' A(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}; \boldsymbol{\sigma}', \hat{\boldsymbol{h}}') + L[\{\hat{P}, \hat{Q}, \boldsymbol{\theta}\}]$$

$$+ i \sum_{\boldsymbol{\sigma}} \int d\hat{\boldsymbol{h}} \, \hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) P(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) + i \sum_{\boldsymbol{\sigma}} \int d\hat{\boldsymbol{h}} \hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) \, Q(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) \quad (20)$$

with

$$A(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}; \boldsymbol{\sigma}', \hat{\boldsymbol{h}}') = Q(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) Q(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') \langle \varepsilon \, e^{-\frac{iJ}{c}(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{h}}' + \boldsymbol{\sigma}' \cdot \hat{\boldsymbol{h}})} \rangle_J$$

$$+ P(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) Q(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') \langle (1 - \varepsilon) \, e^{-\frac{iJ}{c}\boldsymbol{\sigma} \cdot \hat{\boldsymbol{h}}'} \rangle_J$$

$$+ Q(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) P(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') \langle (1 - \varepsilon) \, e^{-\frac{iJ}{c}\boldsymbol{\sigma}' \cdot \hat{\boldsymbol{h}}} \rangle_J \quad (21)$$

and

$$L[\{\hat{P}, \hat{Q}, \boldsymbol{\theta}\}] = \frac{1}{N} \sum_{i=1}^{N} \ln \sum_{\boldsymbol{\sigma}} p_0[\boldsymbol{\sigma}(0)] \int \left(\prod_{t=0}^{t_m-1} \frac{dh(t) \, d\hat{h}(t)}{2\pi} \frac{e^{i\hat{h}(t)[h(t) - \theta_i(t)] + \beta\sigma(t+1)h(t)}}{2\cosh[\beta h(t)]}\right)$$

$$\times e^{-i\hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})} \frac{(-i)^{k_i}}{k_i!} [\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})]^{k_i}. \quad (22)$$

In the above expressions we have neglected those terms that will vanish for $N \to \infty$, and we have removed the now redundant generating fields $\boldsymbol{\psi}$. Upon applying the law of large numbers, the term $L[\{\hat{P}, \hat{Q}, \boldsymbol{\theta}\}]$ simplifies to

$$L[\{\hat{P}, \hat{Q}, \boldsymbol{\theta}\}] = \sum_{k=0}^{\infty} p(k) \ln \sum_{\boldsymbol{\sigma}} p_0[\boldsymbol{\sigma}(0)] \int \left(\prod_{t=0}^{t_m-1} \frac{dh(t) \, d\hat{h}(t)}{2\pi} \frac{e^{i\hat{h}(t)[h(t) - \theta(t)] + \beta\sigma(t+1)h(t)}}{2\cosh[\beta h(t)]}\right)$$

$$\times e^{-i\hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})} \frac{(-i)^k}{k!} [\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})]^k. \quad (23)$$

Functional extremization of $\Psi[\{P, \hat{P}, Q, \hat{Q}\}]$ with respect to the kernels P, $\hat{P}$, Q and $\hat{Q}$, i.e. working out the equations $\delta\Psi/\delta P = \delta\Psi/\delta\hat{P} = \delta\Psi/\delta Q = \delta\Psi/\delta\hat{Q} = 0$, gives the following functional saddle-point equations:

$$\hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) = \mathrm{i}c \sum_{\boldsymbol{\sigma}'} \int \mathrm{d}\hat{\boldsymbol{h}}' Q(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') \langle (1-\varepsilon) \, \mathrm{e}^{-\frac{\mathrm{i}J}{c}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{h}}'} \rangle_J, \tag{24}$$

$$P(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') = \sum_{k=0}^{\infty} p(k) \langle \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \delta(\hat{\boldsymbol{h}} - \hat{\boldsymbol{h}}') \rangle_{\boldsymbol{\theta},k}, \tag{25}$$

$$\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) = \mathrm{i}c \sum_{\boldsymbol{\sigma}'} \int \mathrm{d}\hat{\boldsymbol{h}}' P(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') \langle (1-\varepsilon) \, \mathrm{e}^{-\frac{\mathrm{i}J}{c}\boldsymbol{\sigma}'\cdot\hat{\boldsymbol{h}}} \rangle_J$$
$$+ \mathrm{i}c \sum_{\boldsymbol{\sigma}'} \int \mathrm{d}\hat{\boldsymbol{h}}' Q(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') \langle \varepsilon \, \mathrm{e}^{-\frac{\mathrm{i}J}{c}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{h}}'+\boldsymbol{\sigma}'\cdot\hat{\boldsymbol{h}})} \rangle_J, \tag{26}$$

$$Q(\boldsymbol{\sigma}', \hat{\boldsymbol{h}}') = \sum_{k=1}^{\infty} k p(k) \left\langle \frac{\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \delta(\hat{\boldsymbol{h}} - \hat{\boldsymbol{h}}')}{-\mathrm{i}\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})} \right\rangle_{\boldsymbol{\theta},k}, \tag{27}$$

with a measure $\langle \cdots \rangle_{\boldsymbol{\theta},k}$, which is defined as

$$\langle f(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) \rangle_{\boldsymbol{\theta},k} = \frac{\sum_{\boldsymbol{\sigma}} \int \mathrm{d}\hat{\boldsymbol{h}} \, f(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) M_k(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}|\boldsymbol{\theta})}{\sum_{\boldsymbol{\sigma}} \int \mathrm{d}\hat{\boldsymbol{h}} \, M_k(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}|\boldsymbol{\theta})}, \tag{28}$$

$$M_k(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}|\boldsymbol{\theta}) = \mathrm{e}^{-\mathrm{i}\hat{P}(\boldsymbol{\sigma},\hat{\boldsymbol{h}})} [-\mathrm{i}\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})]^k p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \int \frac{\mathrm{d}h(t)}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\hat{h}(t)[h(t)-\theta(t)]+\beta\sigma(t+1)h(t)}}{2\cosh[\beta h(t)]}. \tag{29}$$

Performing an inverse Fourier transformation of $P(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})$ and $Q(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})$ gives, respectively,

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}') \equiv \int \mathrm{d}\hat{\boldsymbol{h}} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}'\cdot\hat{\boldsymbol{h}}} P(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) = \sum_{k=0}^{\infty} p(k) \langle \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \rangle_{\boldsymbol{\theta}+\boldsymbol{\theta}',k}, \tag{30}$$

$$Q(\boldsymbol{\sigma}|\boldsymbol{\theta}') \equiv \int \mathrm{d}\hat{\boldsymbol{h}} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}'\cdot\hat{\boldsymbol{h}}} Q(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) = \sum_{k=1}^{\infty} k p(k) \left\langle \frac{\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}}{-\mathrm{i}\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})} \right\rangle_{\boldsymbol{\theta}+\boldsymbol{\theta}',k}. \tag{31}$$

The order parameter $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ is the disorder-averaged probability of finding a single-spin trajectory $\boldsymbol{\sigma}$ with modified external fields $\boldsymbol{\theta} + \boldsymbol{\theta}'$ (consisting of the original external fields $\boldsymbol{\theta}$ and the supplement $\boldsymbol{\theta}'$). We will consider the meaning of the order parameter $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ later. The present situation is similar to that in the related studies [41, 44]. In terms of $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ and $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$, the order parameter functions $\hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})$ and $\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}})$ can be rewritten as

$$\hat{P}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) = \mathrm{i}c(1-\varepsilon) \tag{32}$$

$$\hat{Q}(\boldsymbol{\sigma}, \hat{\boldsymbol{h}}) = \mathrm{i}c(1-\varepsilon) \sum_{\boldsymbol{\sigma}'} \langle \mathrm{e}^{-\frac{\mathrm{i}J}{c}\boldsymbol{\sigma}'\cdot\hat{\boldsymbol{h}}} P(\boldsymbol{\sigma}'|\mathbf{0}) \rangle_J + \mathrm{i}c\varepsilon \sum_{\boldsymbol{\sigma}'} \left\langle \mathrm{e}^{-\frac{\mathrm{i}J}{c}\boldsymbol{\sigma}'\cdot\hat{\boldsymbol{h}}} Q\left(\boldsymbol{\sigma}'\Big|\frac{J}{c}\boldsymbol{\sigma}\right) \right\rangle_J. \tag{33}$$

Here we have also used the fact that $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ and $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ are both normalized with respect to summation over $\boldsymbol{\sigma}$, a property which is demonstrated in the next section.

After substituting (32), (33) into (30), (31) and integrating over $\hat{\boldsymbol{h}}$, we finally arrive at the following compact closed-form equations:

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int \mathrm{d}J_\ell \, \tilde{p}(J_\ell) \sum_{\boldsymbol{\sigma}_\ell} \left[\varepsilon Q\left(\boldsymbol{\sigma}_\ell\Big|\frac{1}{c}J_\ell\boldsymbol{\sigma}\right) + (1-\varepsilon)P(\boldsymbol{\sigma}_\ell|\mathbf{0}) \right] \right)$$
$$\times p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \frac{\exp\left(\beta\sigma(t+1)\left[\theta(t) + \theta'(t) + \frac{1}{c}\sum_{\ell=1}^{k} J_\ell\sigma_\ell(t)\right]\right)}{2\cosh\left(\beta\left[\theta(t) + \theta'(t) + \frac{1}{c}\sum_{\ell=1}^{k} J_\ell\sigma_\ell(t)\right]\right)} \tag{34}$$

$$Q(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \sum_{k=0}^{\infty} \frac{k+1}{c} p(k+1) \left(\prod_{\ell=1}^{k} \int \mathrm{d}J_\ell \, \tilde{p}(J_\ell) \sum_{\boldsymbol{\sigma}_\ell} \left[\varepsilon Q\left(\boldsymbol{\sigma}_\ell \Big| \frac{1}{c} J_\ell \boldsymbol{\sigma} \right) + (1-\varepsilon) P(\boldsymbol{\sigma}_\ell|\mathbf{0}) \right] \right)$$

$$\times \, p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \frac{\exp\left(\beta \sigma(t+1)\left[\theta(t) + \theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell(t) \right] \right)}{2\cosh\left(\beta\left[\theta(t) + \theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell(t) \right] \right)}. \tag{35}$$

The equation for the order parameter $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ contains the degree distribution in its bare form $p(k)$. The equation for $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$, on the other hand, involves the deformed degree measure $\frac{k+1}{c}p(k+1)$. This is the only difference between the two expressions. These above two equations are closed and exact and completely general; they cannot be simplified further without making specific parameter choices or assumptions.

In the case of the Poissonian graph [44], the result looks like being represented in terms of only the local field distribution. However, in general both the local field and the cavity field are needed to represent the macroscopic dynamics. In the next section, we discuss the meaning of equations (34) and (35) in detail.

3.2. Physical meaning of order parameter functions and interpretation of closed-form equations

Starting from (15) and (16) and using manipulations similar to those used in deriving (22), we can infer the physical meaning of $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ and $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ at the saddle point. This gives

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}')|_{\text{saddle}} = \langle P(\boldsymbol{\sigma}|\boldsymbol{\theta}') \rangle_* = \frac{1}{N} \sum_{i=1}^{N} \langle \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i} \rangle_* \Big|_{\boldsymbol{\theta}_i \to \boldsymbol{\theta}_i + \boldsymbol{\theta}'}, \tag{36}$$

$$Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')|_{\text{saddle}} = \langle Q(\boldsymbol{\sigma}|\boldsymbol{\theta}') \rangle_* = \frac{1}{N} \sum_{i=1}^{N} \langle \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i} \rangle_* \Big|_{k_i \to k_i-1, \boldsymbol{\theta}_i \to \boldsymbol{\theta}_i + \boldsymbol{\theta}'}, \tag{37}$$

where the brackets $\langle \cdots \rangle_*$ denote evaluation of the argument for the microscopic process (1), i.e.

$$\langle \cdots \rangle_* = \frac{\int \{\mathrm{d}P \, \mathrm{d}\hat{P} \, \mathrm{d}Q \, \mathrm{d}\hat{Q}\} \, \mathrm{e}^{N\Psi[\{P,\hat{P},Q,\hat{Q}\}]}(\cdots)}{\int \{\mathrm{d}P \, \mathrm{d}\hat{P} \, \mathrm{d}Q \, \mathrm{d}\hat{Q}\} \, \mathrm{e}^{N\Psi[\{P,\hat{P},Q,\hat{Q}\}]}}. \tag{38}$$

Details on the derivation of (36), (37) can be found in appendix B. We may now conclude that $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ represents the disorder-averaged probability of finding a single-spin trajectory $\boldsymbol{\sigma}$ in the system from which one site is removed randomly, and in which the external field paths of spins i previously connected to the removed site are modified to $\boldsymbol{\theta}_i + \boldsymbol{\theta}'$. Similarly, $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ represents the disorder-averaged probability of finding a single-spin trajectory $\boldsymbol{\sigma}$ in the system in which the external field paths of a randomly drawn site are modified to $\boldsymbol{\theta}_i + \boldsymbol{\theta}'$ (without removing sites).

Our present general equations (34), (35) have a clearer interpretation than those obtained for strictly Poissonian graphs [44]; in the latter graphs the distinction between (36) and (37) is invisible as a consequence of the property $\frac{k+1}{c}p(k+1) = p(k)$ of the Poissonian degree distribution. To calculate the probability of a single-site path $\boldsymbol{\sigma}$, a random number k is firstly drawn from the degree distribution $p(k)$. This parameter k denotes the number of sites that contribute to the field at the central site. All k associated spin paths of the attached sites $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k$ are sampled from their respective distributions $\varepsilon Q\left(\boldsymbol{\sigma}_\ell | \frac{1}{c} J_\ell \boldsymbol{\sigma}\right) + (1-\varepsilon) P(\boldsymbol{\sigma}_\ell|\mathbf{0})$, which depend on the central site's path $\boldsymbol{\sigma}$. By definition, each site ℓ contributes to the field of the central site. Hence with the probability ε the central site will contribute $\frac{1}{c} J_\ell \boldsymbol{\sigma}$ to the field

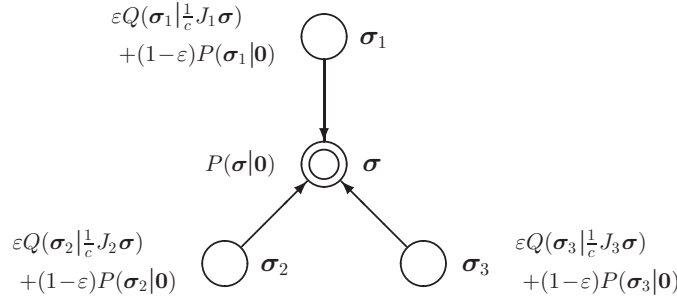**Figure 1.** Illustration of the interpretation of our closed-form order parameter equations, showing how the measure for the spin paths $\boldsymbol{\sigma}$ of a central site with three incoming links is related to the measures of the connected sites under modified conditions. Each connected site has, with probability ε, an incoming link from the central site. For $N \to \infty$ our graphs are locally tree-like; hence the three sites $i = 1, 2, 3$ are mutually correlated only via the central site.

paths of site ℓ, hence the term $\varepsilon Q\big(\boldsymbol{\sigma}_\ell\big|\frac{1}{c}J_\ell\boldsymbol{\sigma}\big)$ in the path measure for ℓ; with the probability $1-\varepsilon$ the central site will not contribute to the fields of site ℓ, hence the term $(1-\varepsilon)P(\boldsymbol{\sigma}_\ell|\mathbf{0})$ in the path measure for ℓ. This process makes it possible to take into account dynamically the effective retarded self-interaction induced by connection symmetry. Figure 1 shows a schematic illustration of this interpretation.

Equations (36) and (37) show explicitly that the order parameters $P(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ and $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$ are normalized, i.e. $\sum_{\boldsymbol{\sigma}} P(\boldsymbol{\sigma}|\boldsymbol{\theta}') = 1$ and $\sum_{\boldsymbol{\sigma}} Q(\boldsymbol{\sigma}|\boldsymbol{\theta}') = 1$. We have used this property to derive (32) and (33). In the special case of Poissonian graphs, i.e. graphs with $p(k) = \mathrm{e}^{-c}c^k/k!$, the degree distribution obeys

$$\frac{k+1}{c}p(k+1) = p(k) \tag{39}$$

and hence one always has $P(\boldsymbol{\sigma}|\boldsymbol{\theta}') = Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$. As a consequence our two closed-form order parameter equations reduce to a single closed equation:

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-c}c^k}{k!} \left(\prod_{\ell=1}^{k} \int \mathrm{d}J_\ell\, \tilde{p}(J_\ell) \sum_{\boldsymbol{\sigma}_\ell} \big[\varepsilon P\big(\boldsymbol{\sigma}_\ell\big|\tfrac{J_\ell}{c}\boldsymbol{\sigma}\big) + (1-\varepsilon)P(\boldsymbol{\sigma}_\ell|\mathbf{0})\big] \right)$$

$$\times\, p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \frac{\exp\big(\beta\sigma(t+1)\big[\theta(t)+\theta'(t)+\frac{1}{c}\sum_{\ell=1}^{k}J_\ell\sigma_\ell(t)\big]\big)}{2\cosh\big(\beta\big[\theta(t)+\theta'(t)+\frac{1}{c}\sum_{\ell=1}^{k}J_\ell\sigma_\ell(t)\big]\big)}. \tag{40}$$

and we thereby recover the results of [44].

4. Fully asymmetric connectivity

4.1. The reduced theory

We first consider the case of fully asymmetric connectivity, i.e. $\varepsilon = 0$. Here the situation is mathematically almost identical to that case of the Poissonian graph [44]. Equation (34) once more closes in terms of $P(\boldsymbol{\sigma}|\mathbf{0})$, and we find

$$P(\boldsymbol{\sigma}|\mathbf{0}) = \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int dJ_\ell \, \tilde{p}(J_\ell) \sum_{\sigma_\ell} P(\boldsymbol{\sigma}_\ell|\mathbf{0}) \right)$$

$$\times \, p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \frac{\exp \left(\beta \sigma(t+1) \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell(t) \right] \right)}{2 \cosh \left(\beta \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell(t) \right] \right)}. \tag{41}$$

Summing both sides of (41) over $\boldsymbol{\sigma}$, except for the entry $\sigma(t+1)$, leads us to single-time spin probabilities $P(\sigma(t+1)|\mathbf{0})$, which are marginal probabilities of the full path measure $P(\boldsymbol{\sigma}|\mathbf{0})$:

$$P(\sigma(t+1)|\mathbf{0}) = \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int dJ_\ell \, \tilde{p}(J_\ell) \sum_{\sigma_\ell(t)} P(\sigma_\ell(t)|\mathbf{0}) \right)$$

$$\times \frac{\exp \left(\beta \sigma(t+1) \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell(t) \right] \right)}{2 \cosh \left(\beta \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell(t) \right] \right)}. \tag{42}$$

It should be noted that the probabilities $P(\sigma(t+1)|\mathbf{0})$ depend only on $P(\sigma(t)|\mathbf{0})$, for any t. This means that there is no effective retarded self-interaction, i.e. there is no effective short loop in the graph with fully asymmetric connectivity. Using the following two general identities:

$$P(\sigma(t)|\mathbf{0}) = \tfrac{1}{2}[1 + \sigma(t)m(t)] \tag{43}$$

$$P(\sigma(t), \sigma(t')|\mathbf{0}) = \tfrac{1}{4}[1 + m(t)\sigma(t) + m(t')\sigma(t') + C(t, t')\sigma(t)\sigma(t')], \tag{44}$$

both the effective single spin magnetization $m(t) = \overline{\langle \sigma(t) \rangle}$ and the covariances $C(t, t') = \overline{\langle \sigma(t)\sigma(t') \rangle}$ can then be written as closed-form iterative expressions respectively:

$$m(t+1) = \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \sum_{\sigma_\ell} \frac{1}{2} [1+\sigma_\ell m(t)] \right) \left\langle \tanh \left(\beta \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell \right] \right) \right\rangle_{J_1, \dots, J_k}, \tag{45}$$

$$C(t+1, t'+1) = \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \sum_{\sigma_\ell, \sigma_\ell'} \frac{1}{4} [1 + m(t)\sigma_\ell + m(t')\sigma_l' + C(t, t')\sigma_\ell \sigma_\ell'] \right)$$

$$\times \left\langle \tanh \left(\beta \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell \right] \right) \tanh \left(\beta \left[\theta(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell' \right] \right) \right\rangle_{J_1, \dots, J_k}, \tag{46}$$

from (42), with $m(0)$ following from the initial conditions. Figure 2 shows a comparison between theory and numerical simulations with respect to the time evolution of the magnetization $m(t)$ on a regular asymmetric sparse graph ($\varepsilon = 0$) with the degree distribution $p(k) = \delta_{k,c}$. The theoretical results are in excellent agreement with the numerical simulations. When taking the limit of $c \to \infty$, the internal fields $v_k(t) = \frac{1}{c} \sum_{l=1}^{k} J_\ell \sigma_\ell$ simplify to $v_k(t) \to \langle J \rangle_J m(t)$. In this limit we therefore have

$$m(t+1) = \tanh(\beta[\theta(t) + \langle J \rangle_J m(t)]), \tag{47}$$

$$C(t, t') = m(t)m(t'). \tag{48}$$

In the absence of external fields, i.e. for $\theta(t) = 0$, a P$\to$ F transition occurs at $\beta = \beta_c$, which is given by $\beta_c \langle J \rangle_J = 1$. This situation is identical to that described in [44].
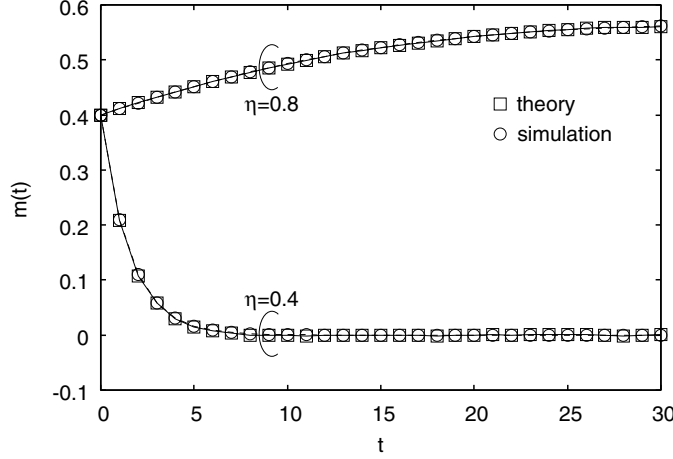
Figure 2. Comparison between theory and numerical simulations with respect to the time evolution of the magnetization $m(t)$ on a asymmetric ($\varepsilon = 0$) finitely connected random graph with a degree distribution $p(k) = \delta_{k,c}$ and zero external fields $\theta(t) = 0$. The initial magnetization is $m(0) = 0.4$. Squares: theoretical results for $c = 3$, $\beta = 3$ and $\eta \in \{0.4, 0.8\}$. Circles: simulation results for $N = 10^5$ spins (averaged over ten runs).

4.2. Time evolution and phase diagrams

We here specialize further and consider the physics described by the $\varepsilon = 0$ equations (45), (46) for the case of having binary random bonds:

$$\tilde{p}(J') = \tfrac{1}{2}(1 + \eta)\delta(J' - J) + \tfrac{1}{2}(1 - \eta)\delta(J' + J), \tag{49}$$

with $\eta \in [-1, 1]$. For such bond statistics the iterative equation (45) for the magnetization reduces to

$$m(t + 1) = \sum_{k=0}^{\infty} p(k) \sum_{r=0}^{k} \binom{k}{r} \left(\frac{1 + \eta m(t)}{2}\right)^r \left(\frac{1 - \eta m(t)}{2}\right)^{k-r} \tanh\left(\beta\left[\theta(t) + \frac{J}{c}(2r - k)\right]\right). \tag{50}$$

For zero external fields, i.e. $\theta(t) = 0$ for all t, and assuming that a stationary state exists, the stationary state magnetizations are given as the solutions of the fixed-point equation $m = F(m)$, with

$$F(m) = \sum_{k=0}^{\infty} p(k) \sum_{r=0}^{k} \binom{k}{r} \left(\frac{1 + \eta m}{2}\right)^r \left(\frac{1 - \eta m}{2}\right)^{k-r} \tanh\left(\frac{\beta J(2r - k)}{c}\right). \tag{51}$$

The map $F(m)$ is anti-symmetric, and hence always has the trivial fixed point $m = 0$. This fixed point is unique for small η, whereas for larger η non-trivial fixed points bifurcate (provided the inverse temperature β is sufficiently large), marking a transition P $\rightarrow$ F from a paramagnetic to a ferro-magnetic state. To determine whether this transition is continuous, and if so find the critical value η_c, we can expand $F(m)$ in powers of m, giving

$$F(m) = \eta m \sum_{k=0}^{\infty} p(k) \sum_{r=0}^{k} \binom{k}{r} \frac{2r - k}{2^k} \tanh\left[\frac{\beta J(2r - k)}{c}\right] + \frac{1}{6}(\eta m)^3 \sum_{k=0}^{\infty} p(k) \sum_{r=0}^{k} \binom{k}{r}$$

$$\times \frac{(2r - k)(4r^2 - 4kr + k^2 - 3k + 2)}{2^k} \tanh\left[\frac{\beta J(2r - k)}{c}\right] + \mathcal{O}(m^5). \tag{52}$$
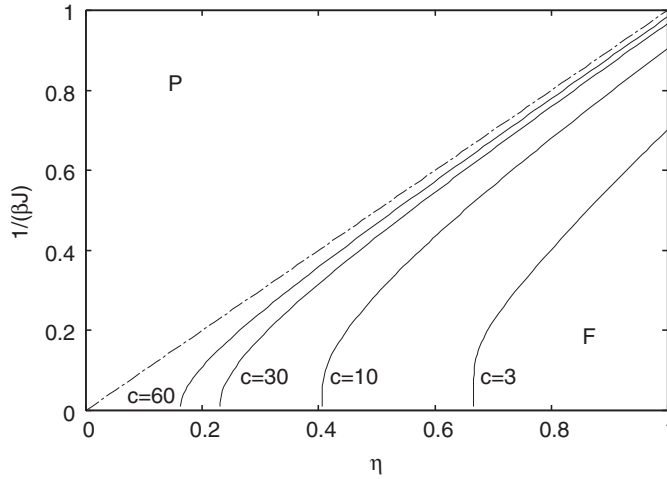
Figure 3. Phase diagram in the $(\eta, 1/(\beta J))$ plane of the $\pm J$ random bond model on a asymmetric ($\varepsilon = 0$) sparse graph with the degree distribution $p(k) = \delta_{k,c}$ and zero external fields $\theta(t) = 0$. Solid lines: P → F transition lines for $c \in \{3, 10, 30, 60\}$. Dashed line: P → F transition line at $1/(\beta J) = \eta$ corresponding to $c = \infty$.

The cubic term can be confirmed to be non-positive. Therefore, there is no evidence for a discontinuous transition, and the critical value η_c can be obtained as

$$\eta_c \sum_{k=0}^{\infty} p(k) \sum_{r=0}^{k} \binom{k}{r} \frac{2r-k}{2^k} \tanh\left[\frac{\beta}{c} J(2r-k)\right] = 1. \tag{53}$$

In a similar way we can inspect the existence and location of a spin-glass phase. Putting $m(t) = 0$ and $\theta(t) = 0$ in (46), one finds that the time-translation invariant covariance $q = \lim_{\tau \to \infty} \lim_{t \to \infty} C(t + \tau, t)$ {is} given as the solutions of $q = G(q)$, with

$$G(q) = \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \sum_{\sigma_\ell, \sigma_\ell'} \frac{1}{4}[1 + \sigma_\ell \sigma_\ell' q]\right) \tanh\left[\frac{\beta J}{c} \sum_{\ell=1}^{k} \sigma_\ell\right] \tanh\left[\frac{\beta J}{c} \sum_{\ell=1}^{k} \sigma_\ell'\right]. \tag{54}$$

It can be confirmed that $G(0) = 0$, $G(q) \leqslant 1$ and $\mathrm{d}^2 G(q)/\mathrm{d}q^2 > 0$ for $q \in [0, 1]$. Hence $G(q) < q$ for $q \in (0, 1]$. Since $q = G(q)$ has no non-trivial solutions, no spin-glass phase exists for $\varepsilon = 0$. Figure 2 shows a comparison between theory and numerical simulations with respect to the time evolution of the magnetization $m(t)$ on a regular asymmetric sparse graph ($\varepsilon = 0$) with the degree distribution $p(k) = \delta_{k,c}$ and zero external fields $\theta(t) = 0$. The theoretical results are in excellent agreement with the numerical simulations. Figure 3 shows the resulting phase diagram in the $(\eta, 1/\beta J)$ plane, for the example of a regular sparse graph with the degree distribution $p(k) = \delta_{k,c}$ and zero external fields $\theta(t) = 0$. For nonzero temperatures, the $c = 2$ regular sparse graph has only a paramagnetic phase (with a zero temperature P → F transition only for $\eta = 1$).

5. Arbitrary connectivity symmetry

5.1. Numerical solution for short times

Equations (34) and (35) are closed and exact, but highly non-trivial, and it is not obvious that they can be simplified further. However, if the bond distribution is of the form (49), the space
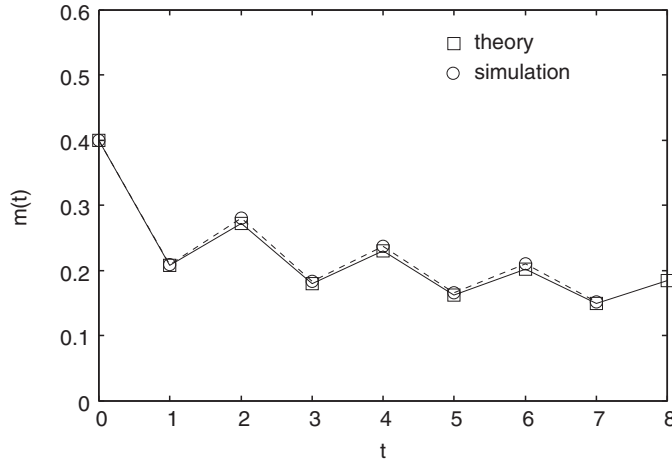
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Figure 4. Comparison between theory and numerical simulations with respect to the time evolution of the magnetization $m(t)$ on a symmetric ($\varepsilon = 1$) finitely connected random graph with a degree distribution $p(k) = \delta_{k,c}$. The initial magnetization is $m(0) = 0.4$. Squares: theoretical results for $c = 3$, $\beta = 3$ and $\eta = 0.4$. Circles: simulation results for $N = 10^5$ spins (averaged over ten runs).

on which our equations are defined at least becomes finite dimensional. In that particular case we can achieve closure for the following reduced order parameters:

$$\mathsf{P}(\boldsymbol{\sigma}) = P(\boldsymbol{\sigma}|\mathbf{0}), \qquad \mathsf{Q}(\boldsymbol{\sigma}|\boldsymbol{\sigma}') = Q\left(\boldsymbol{\sigma}\left|\frac{J}{c}\boldsymbol{\sigma}'\right.\right), \qquad (55)$$

since from equations (34), (35) one can extract

$$\mathsf{P}(\boldsymbol{\sigma}) = \sum_{k=0}^{\infty} p(k)\left(\prod_{\ell=1}^{k} \sum_{\tau_\ell=\pm 1} \frac{1}{2}[1 + \eta\tau_\ell] \sum_{\boldsymbol{\sigma}_\ell} [\varepsilon \mathsf{Q}(\boldsymbol{\sigma}_\ell|\tau_\ell\boldsymbol{\sigma}) + (1-\varepsilon)\mathsf{P}(\boldsymbol{\sigma}_\ell)]\right)$$

$$\times p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \frac{\exp\left(\beta\sigma(t+1)\left[\theta(t) + \frac{J}{c}[\sigma'(t) + \sum_{\ell=1}^{k}\tau_\ell\sigma_\ell(t)]\right]\right)}{2\cosh\left(\beta\left[\theta(t) + \frac{J}{c}[\sigma'(t) + \sum_{\ell=1}^{k}\tau_\ell\sigma_\ell(t)]\right]\right)}, \qquad (56)$$

$$\mathsf{Q}(\boldsymbol{\sigma}|\boldsymbol{\sigma}') = \sum_{k=0}^{\infty} \frac{k+1}{c} p(k+1)\left(\prod_{\ell=1}^{k} \sum_{\tau_\ell=\pm 1} \frac{1}{2}[1 + \eta\tau_\ell] \sum_{\boldsymbol{\sigma}_\ell} [\varepsilon \mathsf{Q}(\boldsymbol{\sigma}_\ell|\tau_\ell\boldsymbol{\sigma}) + (1-\varepsilon)\mathsf{P}(\boldsymbol{\sigma}_\ell)]\right)$$

$$\times p_0[\sigma(0)] \prod_{t=0}^{t_m-1} \frac{\exp\left(\beta\sigma(t+1)\left[\theta(t) + \frac{J}{c}[\sigma'(t) + \sum_{\ell=1}^{k}\tau_\ell\sigma_\ell(t)]\right]\right)}{2\cosh\left(\beta\left[\theta(t) + \frac{J}{c}[\sigma'(t) + \sum_{\ell=1}^{k}\tau_\ell\sigma_\ell(t)]\right]\right)}, \qquad (57)$$

with $\boldsymbol{\sigma}' \in \{-1, 1\}^{t_m}$. Equations (56) and (57) can be solved numerically by iteration. The reduced order parameter function $\mathsf{P}(\boldsymbol{\sigma})$ is the disorder-averaged probability of finding a single-spin trajectory $\boldsymbol{\sigma} = (\sigma(0), \ldots, \sigma(t), \ldots, \sigma(t_m))$ with zero external fields. Therefore, the magnetization is given by

$$m(t) = \sum_{\boldsymbol{\sigma}} \sigma(t)\mathsf{P}(\boldsymbol{\sigma}). \qquad (58)$$

Figure 4 shows a comparison between theory and numerical simulations with respect to the time evolution of the magnetization $m(t)$ on a regular symmetric sparse graph, i.e. $\varepsilon = 1$ and the degree distribution is $p(k) = \delta_{k,c}$. The theoretical results are in good agreement with the

numerical simulations. In the case $\varepsilon = 1$, the numerical analysis in fact simplifies slightly: since (57) no longer involves $\mathsf{P}(\boldsymbol{\sigma})$, one can first solve (57) for $\mathsf{Q}(\boldsymbol{\sigma}|\boldsymbol{\sigma}')$, and then substitute the solution into (56) to obtain $\mathsf{P}(\boldsymbol{\sigma})$.

5.2. Approximate stationary solutions for fully symmetric connectivity

We next consider approximate stationary solutions of our macroscopic equations. Detailed balance holds only for $\varepsilon = 1$. Since we here discuss dynamics with parallel updates, the equilibrium state for $\varepsilon = 1$ does not have a Boltzmann form but involves Peretto's pseudo-Hamiltonian [52]. In the equilibrium state initial conditions are required to be irrelevant; therefore, we may shift the initial and final times to $-\infty$ and ∞, respectively. We also choose zero external fields, $\theta(t) = 0$. To find approximate stationary solutions, we propose the following ansatz:

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \int \mathrm{d}h\, \mathcal{P}(h) \prod_t \frac{\mathrm{e}^{\beta\sigma(t+1)[h+\theta'(t)]}}{2\cosh(\beta[h+\theta'(t)])}, \tag{59}$$

$$Q(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \int \mathrm{d}h\, \mathcal{Q}(h) \prod_t \frac{\mathrm{e}^{\beta\sigma(t+1)[h+\theta'(t)]}}{2\cosh(\beta[h+\theta'(t)])}, \tag{60}$$

where $\mathcal{P}(h)$ denotes an effective local field distribution, and $\mathcal{Q}(h)$ denotes an effective cavity field distribution [4]. Substituting this ansatz into the right-hand sides of (34) and (35) results for large c in (see appendix C for details)

$$
\begin{aligned}
P(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \int \mathrm{d}h \bigg\{ &\sum_{k=0}^{\infty} p(k) \bigg(\prod_{\ell=1}^{k} \int \mathrm{d}h_\ell\, \mathrm{d}J_\ell\, \tilde{p}(J_\ell)\mathcal{Q}(h_\ell) \bigg) \\
&\times \delta\bigg(h - \frac{1}{\beta} \sum_{\ell=1}^{k} \tanh^{-1}\bigg[(\tanh\beta h_\ell)\bigg(\tanh\frac{\beta J}{c} \bigg) \bigg] \bigg) \bigg\} \\
&\times \prod_t \frac{\mathrm{e}^{\beta\sigma(t+1)[h+\theta'(t)]}}{2\cosh(\beta[h+\theta'(t)])}
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
Q(\boldsymbol{\sigma}|\boldsymbol{\theta}') = \int \mathrm{d}h \bigg\{ &\sum_{k=0}^{\infty} \frac{k+1}{c} p(k+1) \bigg(\prod_{\ell=1}^{k} \int \mathrm{d}h_\ell\, \mathrm{d}J_\ell\, \tilde{p}(J_\ell)\mathcal{Q}(h_\ell) \bigg) \\
&\times \delta\bigg(h - \frac{1}{\beta} \sum_{\ell=1}^{k} \tanh^{-1}\bigg[(\tanh\beta h_\ell)\bigg(\tanh\frac{\beta J}{c} \bigg) \bigg] \bigg) \bigg\} \\
&\times \prod_t \frac{\mathrm{e}^{\beta\sigma(t+1)[h+\theta'(t)]}}{2\cosh(\beta[h+\theta'(t)])}
\end{aligned}
\tag{62}
$$

Comparing (61), (59) to (62), (60) then implies the following relationships for the effective field distributions $\mathcal{P}(h)$ and $\mathcal{Q}(h)$:

$$
\begin{aligned}
\mathcal{P}(h) = \sum_{k=0}^{\infty} p(k) \bigg(&\prod_{\ell=1}^{k} \int \mathrm{d}h_\ell\, \mathrm{d}J_\ell\, \tilde{p}(J_\ell)\mathcal{Q}(h_\ell) \bigg) \\
&\times \delta\bigg(h - \frac{1}{\beta} \sum_{\ell=1}^{k} \tanh^{-1}\bigg[(\tanh\beta h_\ell)\bigg(\tanh\frac{\beta J}{c} \bigg) \bigg] \bigg),
\end{aligned}
\tag{63}
$$

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$$
\mathcal{Q}(h) = \sum_{k=0}^{\infty} \frac{k+1}{c} p(k+1) \left(\prod_{\ell=1}^{k} \int \mathrm{d}h_\ell \, \mathrm{d}J_\ell \, \tilde{p}(J_\ell) \mathcal{Q}(h_\ell) \right)
$$

$$
\times \delta \left(h - \frac{1}{\beta} \sum_{\ell=1}^{k} \tanh^{-1} \left[(\tanh \beta h_\ell) \left(\tanh \frac{\beta J}{c} \right) \right] \right). \tag{64}
$$

Equation (64) has a closed form with respect to $\mathcal{Q}(h)$; its solution is substituted into (63) to generate $\mathcal{P}(h)$. In the special case of Poissonian graphs, i.e. $p(k) = \mathrm{e}^{-c} c^k / k!$, we observe that $\mathcal{P}(h) = \mathcal{Q}(h)$, and we therefore have just one closed equation:

$$
\mathcal{P}(h) = \sum_{k=0} \frac{\mathrm{e}^{-c} c^k}{k!} \left(\prod_{\ell=1}^{k} \int \mathrm{d}h_\ell \, \mathrm{d}J_\ell \, \tilde{p}(J_\ell) \mathcal{P}(h_\ell) \right)
$$

$$
\times \delta \left(h - \frac{1}{\beta} \sum_{\ell=1}^{k} \tanh^{-1} \left[\left(\tanh \beta h_\ell \right) \left(\tanh \frac{\beta J}{c} \right) \right] \right). \tag{65}
$$

We thereby recover for Poissonnian graphs the result of [44], which is, in turn, identical to the replica symmetric equilibrium solution of the sequential dynamics version of the model addressed here [5]. In [35], it has been shown that one expects the replica symmetric equilibrium solutions of sequential and parallel dynamics to be identical.

6. Discussion

In this paper, we have applied the generating functional analysis technique to the dynamics of Ising spin models on a finitely connected random graph with arbitrary degree distributions, following the footsteps of [44] (which was limited to Poissonnian degree distributions). We have first derived general exact equations to represent the dynamical properties of the system in the infinite size limit. The introduction of arbitrary degree distributions was found to give us very clear and intuitive interpretations of the macroscopic dynamics of finitely connected Ising systems, in terms of macroscopic path probability distributions involving the evolution of both the actual local field and the cavity field. We have already applied our theory to error-correcting codes [49]. We next aim to apply the theory to other problems in the field of information theory, as well as generalize it to include more complicated local field definitions which will enable it to be used for the analysis of the dynamics of gene regulation systems. Especially in the latter systems, where the graphs concerned are finitely connected and directed but where the degree distributions are known to be far from Poissonnian, the ability to study dynamics macroscopically for arbitrary degree distributions is vital.

Acknowledgments

Appendix A. Calculation of the disorder average

We here calculate the disorder average. The term containing the disorder, which is the left-hand side of (14), becomes

$$
\overline{\exp\left[-\frac{i}{c}\sum_{i=1}^{N}\sum_{t=0}^{t_m-1}\hat{h}_i(t)\sum_{j\neq i}^{N}c_{ij}J_{ij}\sigma_j(t)\right]}
$$

$$
=\frac{1}{Z_c}\sum_{\boldsymbol{c}}\left(\prod_{i=1}^{N}\prod_{j>i}^{N}\hat{p}(c_{ij})\hat{p}(c_{ji}|c_{ij})\int \mathrm{d}J_{ij}\tilde{p}(J_{ij})\right.
$$

$$
\times\exp\left[-\frac{i}{c}\sum_{t=0}^{t_m-1}[\hat{h}_i(t)c_{ij}J_{ij}\sigma_j(t)+\hat{h}_j(t)c_{ji}J_{ji}\sigma_i(t)]\right]\Bigg)
$$

$$
\times\left(\prod_{i=1}^{N}\int_0^{2\pi}\frac{\mathrm{d}\omega_i}{2\pi}\exp\left[i\omega_i\left(k_i-\sum_{j=1}^{N}c_{ij}\right)\right]\right)
$$

$$
=\frac{1}{Z_c}\left(\prod_{i=1}^{N}\int_0^{2\pi}\frac{\mathrm{d}\omega_i}{2\pi}\mathrm{e}^{i\omega_i k_i}\right)\left(\prod_{i=1}^{N}\prod_{j>i}^{N}\left\langle\sum_{c_{ij},c_{ji}}\hat{p}(c_{ij})\hat{p}(c_{ji}|c_{ij})\right.\right.
$$

$$
\times\exp\left[-i\frac{J}{c}\sum_{t=0}^{t_m-1}[\hat{h}_i(t)c_{ij}\sigma_j(t)+\hat{h}_j(t)c_{ji}\sigma_i(t)]-i(\omega_i c_{ij}+\omega_j c_{ji})\right]\Bigg\rangle_J\Bigg).
$$

$$
\tag{A.1}
$$

We then have (14).

Appendix B. Derivation of the physical meaning of our order parameters

The physical meaning of the order parameter functions can be inferred by evaluating (36) and (37). We first present the outline of the derivation of the physical meaning of $Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')$. Equation (37) becomes

$$
Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')|_{\mathrm{saddle}}=\langle Q(\boldsymbol{\sigma}|\boldsymbol{\theta}')\rangle_*
$$

$$
=\frac{\int\{\mathrm{d}P\,\mathrm{d}\hat{P}\,\mathrm{d}Q\,\mathrm{d}\hat{Q}\}\,\mathrm{e}^{N\Psi[\{P,\hat{P},Q,\hat{Q}\}]}\left(\int \mathrm{d}\hat{h}\,\mathrm{e}^{-i\boldsymbol{\theta}'\cdot\hat{h}}\frac{1}{N}\sum_{i=1}^{N}\delta_{\boldsymbol{\sigma},\sigma_i}\delta(\hat{h}-\hat{h}_i)\,\mathrm{e}^{-i\omega_i}\right)}{\int\{\mathrm{d}P\,\mathrm{d}\hat{P}\,\mathrm{d}Q\,\mathrm{d}\hat{Q}\}\,\mathrm{e}^{N\Psi[\{P,\hat{P},Q,\hat{Q}\}]}}
$$

$$
=\left[\int\{\mathrm{d}P\,\mathrm{d}\hat{P}\,\mathrm{d}Q\,\mathrm{d}\hat{Q}\}\,\mathrm{e}^{N\Psi[\{P,\hat{P},Q,\hat{Q}\}]}\right]^{-1}
$$

$$
\times\left[\frac{1}{N}\sum_{i=1}^{N}\int\{\mathrm{d}P\,\mathrm{d}\hat{P}\,\mathrm{d}Q\,\mathrm{d}\hat{Q}\}\,\mathrm{e}^{N\Omega[\{P,\hat{P},Q,\hat{Q}\}]}\right.
$$

$$
\times\left\{\prod_{j=1}^{N}\left(\sum_{\sigma_j}\int \mathrm{d}\boldsymbol{h}_j\,\mathrm{d}\hat{\boldsymbol{h}}_j\int \mathrm{d}\omega_j\,\mathrm{e}^{i\omega_j k_j}\rho_j(\boldsymbol{\sigma}_j,\boldsymbol{h}_j,\hat{\boldsymbol{h}}_j,\boldsymbol{\theta}_i,\omega_j)\right)\right\}
$$

$$
\times\int \mathrm{d}\hat{\boldsymbol{h}}\,\mathrm{e}^{-i\boldsymbol{\theta}'\cdot\hat{h}}\delta_{\boldsymbol{\sigma},\sigma_i}\delta(\hat{h}-\hat{h}_i)\,\mathrm{e}^{-i\omega_i}\right],
$$

$$
\tag{B.1}
$$

where $\rho_i(\boldsymbol{\sigma}_i,\boldsymbol{h}_i,\hat{\boldsymbol{h}}_i,\boldsymbol{\theta}_i,\omega_i)$ and $\Omega(\{P,\hat{P},Q,\hat{Q}\})$ are defined as follows:

$$
\rho_i(\boldsymbol{\sigma}_i,\boldsymbol{h}_i,\hat{\boldsymbol{h}}_i,\boldsymbol{\theta}_i,\omega_i)=\frac{1}{(2\pi)^{t_m+1}}\,\mathrm{e}^{-i\hat{P}(\boldsymbol{\sigma}_i,\hat{h}_i)}\,\mathrm{e}^{-i\hat{Q}(\boldsymbol{\sigma}_i,\hat{h}_i)\,\mathrm{e}^{-i\omega_i}}
$$

$$\times\, p_0[\sigma_i(0)]\prod_{t=0}^{t_m-1}\frac{\mathrm{e}^{\mathrm{i}\hat{h}_i(t)[h_i(t)-\theta_i(t)]+\beta\sigma_i(t+1)h_i(t)}}{2\cosh[\beta h_i(t)]},\tag{B.2}$$

$$\Omega[\{P,\hat{P},Q,\hat{Q}\}]=\frac{c}{2}(\varepsilon-2)+\frac{c}{2}\sum_{\sigma}\sum_{\sigma'}\int\mathrm{d}\hat{h}\,\mathrm{d}\hat{h}'A(\sigma,\hat{h};\sigma',\hat{h}')$$

$$+\mathrm{i}\sum_{\sigma}\int\mathrm{d}\hat{h}[\hat{P}(\sigma,\hat{h})P(\sigma,\hat{h})+\hat{Q}(\sigma,\hat{h})Q(\sigma,\hat{h})].\tag{B.3}$$

The term in (B.1) that involves $\boldsymbol{\omega}$-integrations can be evaluated as follows:

$$\left\{\prod_{j=1}^{N}\left(\sum_{\sigma_j}\int\mathrm{d}h_j\,\mathrm{d}\hat{h}_j\,\mathrm{d}\omega_j\,\mathrm{e}^{\mathrm{i}\omega_j k_j}\rho_j(\sigma_j,h_j,\hat{h}_j,\theta_i,\omega_j)\right)\right\}\int\mathrm{d}\hat{h}\,\mathrm{e}^{-\mathrm{i}\theta'\cdot\hat{h}}\delta_{\sigma,\sigma_i}\delta(\hat{h}-\hat{h}_i)\,\mathrm{e}^{-\mathrm{i}\omega_i}$$

$$=\left\{\prod_{j\neq i}^{N}\left(\sum_{\sigma_j}\int\mathrm{d}h_j\,\mathrm{d}\hat{h}_j\,\mathrm{d}\omega_j\,\mathrm{e}^{\mathrm{i}\omega_j k_j}\rho_j(\sigma_j,h_j,\hat{h}_j,\theta_i,\omega_j)\right)\right\}$$

$$\times\sum_{\sigma_i}\int\mathrm{d}h_i\mathrm{d}\hat{h}\,\mathrm{d}\omega_i\,\mathrm{e}^{\mathrm{i}\omega_i(k_i-1)}\rho_i(\sigma_i,h_i,\hat{h},\theta_i+\theta',\omega_i)\delta_{\sigma,\sigma_i}$$

$$=\left\{\prod_{j=1}^{N}\left(\sum_{\sigma_j}\int\mathrm{d}h_j\mathrm{d}\hat{h}_j\,\mathrm{d}\omega_j\,\mathrm{e}^{\mathrm{i}\omega_j k_j}\rho_j(\sigma_j,h_j,\hat{h}_j,\theta_i,\omega_j)\right)\right\}\Bigg|_{k_i\to k_i-1,\theta_i\to\theta_i+\theta'}.\tag{B.4}$$

Substituting (B.4) into (B.1) then gives us (37). Equation (36) can be obtained in a similar way, since the difference is just the absence of the factor $\mathrm{e}^{-\mathrm{i}\omega_i}$.

Appendix C. Derivation of approximate stationary solutions

Equations (61) and (62) are obtained following the reasoning in [44]. We first derive (61), using two specific identities. The first applies to $\sigma\in\{-1,1\}$:

$$\cosh[\beta(a+b\sigma)]=A\,\mathrm{e}^{\beta B\sigma},\tag{C.1}$$

with

$$A=\sqrt{\cosh[\beta(a+b)]\cosh[\beta(a-b)]},\qquad B=\frac{1}{2\beta}\ln\frac{\cosh[\beta(a+b)]}{\cosh[\beta(a-b)]}.\tag{C.2}$$

The second identity holds for $\sigma\in\{-1,0,1\}$ and any function $f(\sigma)$:

$$f(\sigma)=C\,\mathrm{e}^{\beta D\sigma+\beta E\sigma^2},\tag{C.3}$$

with

$$C=f(0),\qquad D=\frac{1}{2\beta}\ln\frac{f(1)}{f(-1)},\qquad E=\frac{1}{2\beta}\ln\frac{f(1)f(-1)}{f(0)^2}.\tag{C.4}$$

We abbreviate $\prod_t\left[\frac{1}{2}\sum_{\sigma(t)=\pm1}f(\sigma)\right]=\langle f(\sigma)\rangle_\sigma$ and substitute the ansatz (60) into the right-hand side of (34) with $\varepsilon=1$, which gives

$$P(\sigma|\theta')=\sum_{k=0}^{\infty}p(k)\Bigg\langle\left(\prod_{\ell=1}^{k}\int\mathrm{d}J_\ell\,\tilde{p}(J_\ell)\int\mathrm{d}h\,\mathcal{Q}(h)\prod_t\frac{\mathrm{e}^{\beta\sigma(t+1)[h+\theta'(t)]}}{\cosh(\beta[h+\theta'(t)])}\right)$$

$$\times\prod_t\frac{\exp\left(\beta\sigma(t+1)\left[\theta'(t)+\frac{1}{c}\sum_{\ell=1}^{k}J_\ell\sigma_\ell(t)\right]\right)}{2\cosh\left(\beta\left[\theta'(t)+\frac{1}{c}\sum_{\ell=1}^{k}J_\ell\sigma_\ell(t)\right]\right)}\Bigg\rangle_\sigma$$

$$= \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int dJ_\ell \, \tilde{p}(J_\ell) \int dh \, \mathcal{Q}(h) \right) \left\{ \prod_t \frac{e^{\beta \sigma(t+1)[\theta'(t) - \sum_{\ell=1}^{k} B_\ell]}}{\prod_{\ell=1}^{k} A_\ell} \right\}$$

$$\times \prod_t \left\langle \frac{\exp\left(\beta \sum_{\ell=1}^{k} \sigma_\ell \left[h_\ell + \frac{1}{c} J_\ell \{\sigma(t-1) + \sigma(t+1)\} \right]\right)}{2 \cosh\left(\beta \left[\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell\right]\right)} \right\rangle_{\sigma_1, \dots, \sigma_k} \tag{C.5}$$

$$= \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int dJ_\ell \, \tilde{p}(J_\ell) \int dh \, \mathcal{Q}(h) \right) \left(\prod_t \frac{C_{k,t} \, e^{\frac{1}{2}\beta E_{k,t}}}{\prod_{\ell=1}^{k} A_\ell} \right)$$

$$\times \prod_t e^{\beta \sigma(t+1)[\theta(t) - \sum_{\ell=1}^{k} B_\ell + \frac{1}{2} D_{k,t} + \frac{1}{2} D_{k,t+2} + \frac{1}{2} F_{k,t} \sigma(t-1)]} \tag{C.6}$$

$$= \int dh \left\{ \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int dh_\ell \, dJ_\ell \, \tilde{p}(J_\ell) \mathcal{Q}(h_\ell) \right) \prod_t \frac{e^{\beta \sigma(t+1)[h + \theta'(t)]}}{2 \cosh\left(\beta [h + \theta'(t)]\right)} \right.$$

$$\left. \times \delta \left(h - \frac{2}{\beta c} \sum_{\ell=1}^{k} J_\ell \tanh \beta J_\ell + \frac{1}{2\beta} \sum_{\ell=1}^{k} \ln \frac{\cosh\left(\beta [h_\ell + \frac{1}{c} J_\ell]\right)}{\cosh(\beta [h_\ell - \frac{1}{c} J_\ell])} \right) \right\} \tag{C.7}$$

$$= \int dh \left\{ \sum_{k=0}^{\infty} p(k) \left(\prod_{\ell=1}^{k} \int dh_\ell \, dJ_\ell \, \tilde{p}(J_\ell) \mathcal{Q}(h_\ell) \right) \prod_t \frac{e^{\beta \sigma(t+1)[h + \theta'(t)]}}{2 \cosh(\beta [h + \theta'(t)])} \right.$$

$$\left. \times \delta \left(h - \frac{1}{\beta} \sum_{\ell=1}^{k} \tanh^{-1} \left[(\tanh \beta h_\ell) \left(\tanh \frac{\beta J}{c} \right) \right] \right) \right\} \tag{C.7}$$

where we have put

$$A_\ell = \sqrt{\cosh\left(\beta \left[h_\ell + \frac{1}{c} J_\ell \right]\right) \cosh\left(\beta \left[h_\ell - \frac{1}{c} J_\ell \right]\right)}, \tag{C.8}$$

$$B_\ell = \frac{1}{2\beta} \ln \frac{\cosh\left(\beta [h_\ell + \frac{1}{c} J_\ell]\right)}{\cosh(\beta [h_\ell - \frac{1}{c} J_\ell])}, \tag{C.9}$$

$$C_{k,t} = \left\langle \frac{\exp\left(\beta \sum_{\ell=1}^{k} \sigma_\ell h_\ell\right)}{2 \cosh\left(\beta \left[\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell\right]\right)} \right\rangle_{\sigma_1, \dots, \sigma_k}, \tag{C.10}$$

$$D_{k,t} = \frac{1}{2\beta} \ln \frac{\left\langle \frac{\exp(\beta \sum_{\ell=1}^{k} \sigma_\ell [h_\ell + \frac{2}{c} J_\ell])}{2 \cosh(\beta [\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell])} \right\rangle_{\sigma_1, \dots, \sigma_k}}{\left\langle \frac{\exp(\beta \sum_{\ell=1}^{k} \sigma_\ell [h_\ell - \frac{2}{c} J_\ell])}{2 \cosh(\beta [\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell])} \right\rangle_{\sigma_1, \dots, \sigma_k}}, \tag{C.11}$$

$$E_{k,t} = \frac{1}{2\beta} \ln \frac{\left\langle \frac{\exp(\beta \sum_{\ell=1}^{k} \sigma_\ell [h_\ell + \frac{2}{c} J_\ell])}{2 \cosh(\beta [\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell])} \right\rangle_{\sigma_1, \dots, \sigma_k} \left\langle \frac{\exp(\beta \sum_{\ell=1}^{k} \sigma_\ell [h_\ell - \frac{2}{c} J_\ell])}{2 \cosh(\beta [\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell])} \right\rangle_{\sigma_1, \dots, \sigma_k}}{\left\langle \frac{\exp(\beta \sum_{\ell=1}^{k} \sigma_\ell h_\ell)}{2 \cosh(\beta [\theta'(t) + \frac{1}{c} \sum_{\ell=1}^{k} J_\ell \sigma_\ell])} \right\rangle_{\sigma_1, \dots, \sigma_k}^2}. \tag{C.12}$$

To obtain (C.5) and (C.6) we have used (C.1) and (C.3), respectively. Equation (62) can be derived in the same way.

References

[1] Viana L and Bray A J 1985 *J. Phys. C: Solid State Phys.* **18** 3037

[2] Kanter I and Sompolinsky H 1987 *Phys. Rev. Lett.* **58** 164

[3] Mezard M and Parisi G 1987 *Europhys. Lett.* **3** 1067
[4] Mézard M, Parisi G and Virasoro M 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[5] Mézard M and Parisi G 1987 *Europhys. Lett.* **3** 1067
[6] Mottishaw P and De Dominicis C 1987 *J. Phys. A: Math. Gen.* **20** L375
[7] Wong K Y and Sherrington D 1988 *J. Phys. A: Math. Gen.* **21** L459
[8] Monasson R 1988 *J. Phys. A: Math. Gen.* **31** 513
[9] Coolen A C C, Skantzos N S, Pérez Castillo I, Pérez Vicente C J, Hatchett J P L, Wemmenhove B and Nikoletopoulos T 2005 *J. Phys. A: Math. Gen.* **38** 8289
[10] Pérez Vicente C J and Coolen A C C 2008 *J. Phys. A: Math. Theor.* **41** 255003
[11] Sourlas N 1989 *Nature* **339** 693
[12] Kabashima Y and Saad D 1999 *Europhys. Lett.* **45** 97
[13] Vicente R, Saad D and Kabashima Y 1999 *Phys. Rev.* E **60** 5352
[14] Murayama T, Kabashima Y, Saad D and Vicente R 2000 *Phys. Rev.* E **62** 1577
[15] Nakamura K, Kabashima Y and Saad D 2001 *Europhys. Lett.* **56** 610
[16] Nishimori H 2001 *Statistical Physics of Spin Glasses and Information Processing* (Oxford: Oxford University Press)
[17] Alamino R C and Saad D 2007 *J. Phys. A: Math. Theor.* **40** 12259
[18] Neri I, Skantzos N S and Bollé D 2008 *J. Stat. Mech.: Theory Exp.* P10018
[19] Matsunaga Y and Yamamoto H 2003 *IEEE Trans. Inf. Theory* **49** 2225
[20] Murayama T and Okada M 2003 *J. Phys. A: Math. Gen.* **36** 11123
[21] Murayama T 2004 *Phys. Rev.* E **69** 035105
[22] Wainwright M J and Maneva E 2005 *Proc. of 2005 Int'l Symp. Info. Theory (ISIT)* p 1493
[23] Ciliberti S and Mézard M 2006 *J. Stat. Mech.* **3** 58
[24] Ciliberti S, Mézard M and Zecchina R 2006 *Complex syst. methods* **3** 58
[25] Martinian E and Wainwright M J 2006 *Workshop on Info. Theory and Its Applications* p 229
[26] Mimura K 2009 *J Phys. A: Math. Theor.* **42** 135002
[27] Yoshida M and Tanaka T 2006 *Proc. of 2006 Int'l Symp. Info. Theory (ISIT)* p 2378
[28] Raymond J and Saad D 2007 *J. Phys. A: Math. Theor.* **40** 12315
[29] Guo D and Wang C 2008 *IEEE J. Sel. Areas Commun.* **26** 421
[30] Kirkpatrick S and Selman B 1994 *Science* **264** 1297
[31] Monasson R and Zecchina R 1998 *Phys. Rev.* E **56** 1357
[32] Monasson R and Zecchina R 1998 *J. Phys. A: Math. Gen.* **31** 9209
[33] Monasson R, Zecchina R, Kirkpatrick S, Selman B and Troyansky L 1999 *Nature* **400** 133
[34] Zdeborová L and Mézard Marc 2008 *Phys. Rev. Lett.* **101** 078702
[35] Pérez Castillo I and Skantzos N S 2004 *J. Phys. A: Math. Gen.* **37** 9087
[36] Pérez Castillo I, Wemmenhove B, Hatchett J P L, Coolen A C C, Skantzos N S and Nikoletopoulos T 2004 *J. Phys. A: Math. Gen.* **37** 8789
[37] Gitterman A 2000 *J. Phys. A: Math. Gen.* **33** 8373
[38] Nikoletopoulos T, Coolen A C C, Pérez Castillo I, Skantzos N S, Hatchett J P L and Wemmenhove B 2004 *J. Phys. A: Math. Gen.* **37** 6455
[39] Hatchett J P L, Skantzos N S and Nikoletopoulos T 2005 *Phys. Rev.* E **72** 066105
[40] Skantzos N S, Pérez Castillo Isaac and Hatchett J P L 2005 *Phys. Rev.* E **72** 066127
[41] Semerjian G and Cugliandolo L F 2003 *Europhys. Lett.* **61** 2, 247
[42] Semerjian G, Cugliandolo L F and Montanari A 2004 *J. Stat. Phys.* **115** 493
[43] Semerjian G and Weigt M 2004 *J. Phys. A: Math. Gen.* **37** 5525
[44] Hatchett J P L, Wemmenhove B, Pérez Castillo I, Nikoletopoulos T, Skantzos N S and Coolen A C C 2004 *J. Phys. A: Math. Gen.* **37** 6201
[45] Hatchett J P L, Pérez Castillo I, Coolen A C C and Skantzos N S 2005 *Phys. Rev. Lett.* **95** 117204
[46] Skantzos N S and Hatchett J P L 2007 *Physica* A **381** 202
[47] Mozeika A and Coolen A C C 2008 *J. Phys. A: Math. Theor.* **41** 115003
[48] Mozeika A and Coolen A C C 2009 *J. Phys. A: Math. Theor.* **42** 195006
[49] Mimura K and Coolen A C C 2009 *Proc. of 2009 Int'l Symp. Info. Theory (ISIT)* p 1829
[50] Neri I and Bollé D 2009 The cavity approach to the parallel dynamics on finitely connected graphs arXiv:0905.3260
[51] De Dominicis C 1978 *Phys. Rev.* B **18** 4913
[52] Peretto P 1984 *Biol. Cybern.* **50** 51